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A method for calculating direct and inverse problems of unsteady-state heat transfer

P. V. TSOI
Tadzhik Politechnic Institute, 734042, Dyushambe, U.S.S.R.

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Abstract—An effective analytical method is suggested for the solution of direct and inverse problems of heat conduction, thermoelastic stresses and heat transfer by singly and doubly taking integral transformations with respect to unilateral parabolic variables and by implementing the method of finite elements of the orthogonal residual projection in a certain functional space over the entire range of variation of bilateral elliptical coordinates.

THE SOLUTION of unsteady-state heat conduction and heat transfer boundary-value problems for heat transfer agent flow in tubes under the conditions of asymmetric thermal loading by applying rigorous analytical methods leads to rather cumbersome mathematical transformations, with the temperature fields being expressed by complex functional series, thus making it difficult to incorporate the solutions obtained into thermal engineering practice. A temperature field can be represented by a simple, reasonably accurate analytical formula which is especially important when being obtained for the intermediate stage in the solution of more complex problems such as, for example, the determination of thermoelastic stresses in the elements of constructions or the study of inverse heat conduction and heat transfer problems. One of these methods is suggested in the present paper. It is based on simultaneous application of integral transformations with respect to unilateral parabolic variables and of the orthogonal residual projection for the differential operator of transfer in a certain functional space over bilateral elliptic coordinates in the terminology of Patankar [1]. The method allows one to find solutions for a wide range of problems in one- and two-dimensional bodies of simple and complex shapes, i.e. to find approximate analytical solutions even in those cases, when exact solutions are impossible.

As is known, in most cases the solutions of hydrodynamics and heat transfer problems by classical methods "contain infinite series, special functions, transcendental equations for eigenvalues etc., and their numerical estimates may present a very formidable problem" [1]. As a rule, only partial sums of several terms from the resulting functional series are used for calculations. The technique suggested in this paper for determining temperature fields makes it possible to find approximate solutions, equivalent to these terms, in the form of polynomials in spatial elliptical coordinates the coefficients of which are exponentially stabilized over parabolic variables (over the time and longitudinal unilateral coordinate). Then for a certain class of entry functions of the temperature perturbation such an approximate solution is found, by using the method of an optimal selection of a system of basic coordinates, the asymptote of which coincides with an exact solution $Fo \to \infty$ and $X \to \infty$ [2]. Note that these limiting solutions cannot be obtained from a partial sum.

The method developed in the course of the solution of the main problem yields rather accurate values for the squared roots of a characteristic equation (of the first and subsequent eigenvalues) without solving the transcendental equations themselves. This is important for studying the processes occurring under regular and quasi-regular conditions in multi-dimensional geometric bodies for which it is impossible to write the transfer equations in explicit characteristic form.

The method will now be presented in relation to the solution of direct and inverse non-stationary problems of heat conduction, thermal stresses and heat transfer during liquid flow in tubes.

(1) The problems of heat conduction in a plate ($\Gamma = 0$), in the wall of a circular tube ($\Gamma = 1$) and in a spherical shell ($\Gamma = 2$), that exchanges heat by convection with media having temperatures $\phi_1(Fo)$ and $\phi_2(Fo)$, admit a single statement

$$\frac{\partial T}{\partial Fo} = \frac{1}{(m\rho+1)^{\Gamma}} \frac{\partial}{\partial \rho}\left((m\rho+1)^{\Gamma} \frac{\partial T}{\partial \rho}\right) + \frac{q_v(\rho, Fo)R^2}{\lambda} \tag{1}$$

$$[T(\rho, Fo)]_{Fo=0} = T_0 \tag{2}$$

$$\left\{\frac{\partial T}{\partial \rho} - Bi_1\, T(\rho, Fo)\right\}_{\rho=0} = -Bi_1\, \phi_1(Fo)$$

$$\left\{\frac{\partial T}{\partial \rho} + Bi_2\, T(\rho, Fo)\right\}_{\rho=1} = Bi_2\, \phi_2(Fo). \tag{3}$$

Equations (1)–(3) involve many typical heat conduction problems for bodies of three classical geometric shapes. For example, when $Bi_1 = 1$, $R_1 = 0$, there is a problem for solid bodies.

NOMENCLATURE

a	thermal diffusivity
Bi	Biot number, $\alpha R/\lambda$
c	heat capacity
E	Young's modulus
Fo	Fourier number, at/R^2
k	R_2/R_1
m	R/R_1
Pd	Predvoditelev number
Po	Pomerantsev number, $(q_v R^2)/\lambda(T_w - T_0)$
r	instantaneous radius
R	wall thickness, $R_2 - R_1$
R_1, R_2	inner and outer radii of a hollow cylinder or spherical shell
X	relative coordinate of a semi-infinite rod, x/R.

Greek symbols

β	coefficient of volumetric expansion
γ	specific density
Γ	geometric shape parameter
λ	thermal conductivity
θ	relative excess temperature
ρ	instantaneous relative coordinate, $(r-R_1)/R$.

Other symbol

$\risingdotseq$	sign of transition to the double integral Laplace transform and Fourier cosine transform region.

Assume that

$$\bar{T}(\rho,p) = \int_0^\infty T(\rho, Fo)\exp(-p\,Fo)\,\mathrm{d}Fo$$

then in the space of the Laplace integral transforms one obtains

$$\mathscr{L}[\bar{T}(\rho,p)] = \frac{\mathrm{d}}{\mathrm{d}\rho}\left((m\rho+1)^\Gamma \frac{\mathrm{d}\bar{T}}{\mathrm{d}\rho}\right) - [p\bar{T}(\rho,p) - T_0](m\rho+1)^\Gamma + \frac{\bar{q}_v(\rho,p)(m\rho+1)^\Gamma R^2}{\lambda} = 0 \quad (4)$$

$$\left\{\frac{\mathrm{d}\bar{T}}{\mathrm{d}\rho} - Bi_1\,\bar{T}(\rho,p)\right\}_{\rho=0} = -Bi_1\,\bar{\Phi}_1(p)$$

$$\left\{\frac{\mathrm{d}\bar{T}}{\mathrm{d}\rho} + Bi_2\,\bar{T}(\rho,p)\right\}_{\rho=1} = Bi_2\,\bar{\Phi}_2(p). \quad (5)$$

By using the method of undetermined coefficients, the linear function $\bar{\Phi}(\rho,p) = A\rho + B$ is found, which satisfies conditions (5), and then

$$\bar{\Phi}(\rho,p) = \frac{Bi_1\,Bi_2[\bar{\Phi}_2(p) - \bar{\Phi}_1(p)]\rho + Bi_2\,\bar{\Phi}_2(p) + (1+Bi_2)Bi_1\,\bar{\Phi}_1(p)}{Bi_1 + Bi_2 + Bi_1\,Bi_2}. \quad (6)$$

The approximate solution of boundary problems (4) and (5) is considered as an element (vector) in the finite-dimensional space and belongs to the set of composition of the form

$$\bar{T}_n(\rho,p) = \bar{\Phi}(\rho,p) + \sum_{k=1}^n \bar{a}_k(p)\psi_k(\rho) \quad (7)$$

where the system of basic coordinates $\{\psi_k(\rho)\}$ satisfies homogeneous boundary conditions (5). The optimal first basic function for the cases

$$\lim_{Fo\to\infty} q_v(\rho, Fo) = q_v = \text{const.}$$

is taken to be

$$\psi_1(\rho) = \frac{(Bi_2+2)(1+Bi_1\,\rho)}{Bi_1 + Bi_2 + Bi_1\,Bi_2} - \rho^2$$

and the rest functions are selected *a priori*, for example

$$\psi_k(\rho) = (1-\rho)^2\rho^k, \quad k \geqslant 2.$$

The coefficient images $\bar{a}_k(p)$ in the presence of which the left-hand side of equation (4) deviates the least from zero over the entire range of ρ are found by requiring that the residual $\mathscr{L}[\bar{T}_n(\rho,p)] \neq 0$ be orthogonal with respect to all basic coordinates $\psi_j(\rho)$

$$\int_0^1 \mathscr{L}[\bar{T}_n(\rho,p)]\psi_j(\rho)\,\mathrm{d}\rho = 0, \quad j = 1,2,\ldots,n.$$

Upon integration with respect to ρ the system is reduced to

$$\left\{\sum_{k=1}^n \bar{a}_k(p)(A_{jk} + B_{jk}p) = \bar{D}_j(p)\right\}, \quad j = 1,2,\ldots,n \quad (8)$$

where

$$A_{jk} = A_{kj} = -\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\rho}\left((m\rho+1)^\Gamma \frac{\mathrm{d}\psi_k}{\mathrm{d}\rho}\right)\psi_j(\rho)\,\mathrm{d}\rho > 0$$

$$B_{jk} = B_{kj} = \int_0^1 (m\rho+1)^\Gamma \psi_j(\rho)\psi_k(\rho)\,\mathrm{d}\rho > 0$$

$$\bar{D}_j(p) = \int_0^1 \mathscr{L}[\bar{\Phi}(\rho,p)]\psi_j(\rho)\,\mathrm{d}\rho. \quad (9)$$

The determination of $\bar{a}_k(p)$ from system (8) by the formula

$$\bar{a}_k(p) = \frac{\sum_{j=1}^{n} D_j(p)\Delta_{jk}}{\Delta(p)} \tag{10}$$

and the transition to the domain of inverted transforms using the convolution theorem [3] yields

$$a_k(Fo) = \sum_{i=1}^{n} \left\{ \int_0^{Fo} \frac{\sum_{j=1}^{n} D_j(\tau)\Delta_{jk}(p_i)\exp[p_i(Fo-\tau)]\,d\tau}{\Delta'(p_i)} \right\} \tag{11}$$

where $\Delta(p) = |A+Bp|$ is the main determinant of system (8); $Pi < 0$ are the simple roots of $\Delta(p) = 0$; $\Delta_{jk}(p)$ are the cofactors of the determinant $\Delta(p)$; $D_j(Fo)$ is the inverted transform $\bar{D}_j(p)$; $\Delta' = d\Delta/dp$.

After transition to the domain of inverted transforms in relation (7) taking into account equation (11), it is possible to find the solution of the problem on compliance with the specific conditions of uniqueness for the entry functions of temperature perturbation

$$\phi_1(Fo), \quad \phi_2(Fo), \quad q_v(\rho, Fo).$$

Hence, the proposed algorithm for the solution of the problem posed requires the setting up of algebraic system (8) and then the conversion by a single equation (11) for all the three bodies of classical shapes, thus enabling one to run all laborious calculations on an electronic computer when obtaining an analytical solution or implementing the difference method.

Let $R_1 = 0$, $Bi_1 = 0$, $Bi_2 = Bi$, $q_v(\rho, Fo) = q_v = \text{const}$. Then the truncated system (8) of first order gives

$$\bar{a}_1(p) = \frac{[T_0 - p\bar{\phi}(p)]A(Bi,\Gamma)}{2(\Gamma+1)[p+A(Bi,\Gamma)]} - \frac{q_v R^2}{2\lambda(\Gamma+1)}\left[\frac{1}{p} - \frac{1}{p+A(Bi,\Gamma)}\right] \tag{12}$$

where

$$A(Bi,\Gamma) = \frac{Bi(\Gamma+1)(\Gamma+5)(Bi+3+\Gamma)}{2Bi^2+2Bi(\Gamma+5)+(\Gamma^2+18\Gamma+15)}. \tag{13}$$

Equation (12) makes it possible to find one unified solution for the three bodies when the variation in the surrounding temperature follows a specific law. When $\phi(Fo) = \text{const.}$, one obtains

$$\theta(\rho, Fo) = \frac{T(\rho, Fo) - T_0}{T_c - T_0} = 1 - \frac{A(Bi,\Gamma)}{2(\Gamma+1)} \times \left(\frac{Bi+2}{Bi} - \rho^2\right)\exp[-A(Bi,\Gamma)Fo] + \frac{Po}{2(\Gamma+1)} \times \{1 - \exp[-A(Bi,\Gamma)Fo]\}\left(\frac{Bi+2}{Bi} - \rho^2\right). \tag{14}$$

When $Fo \to \infty$, this yields an exact solution of the stationary problem. The quantity $A(Bi,\Gamma)$ approximates, with good accuracy, the dependence of the first squared roots on Bi of the three characteristic equations for a plate, cylinder and a sphere. The solution of the truncated system (8) of second order may give a more precise equation, than relation (13), and also an expression for the second eigenvalue. These investigations and comparisons are discussed in ref. [2]. The uncertainties of the aforegoing equations increase monotonously with Bi and at $Bi = \infty$ attain the greatest value not exceeding 3–5%.

In order to implement the algorithm of the method on electronic digital computers, the solutions in the third and subsequent approximations should be obtained at the given values of Bi, Γ. The results of calculations of the eigenvalues for a cylinder ($\Gamma = 1$) under the first-kind boundary conditions ($Bi = \infty$) by solving the equation $\Delta(p) = 0$ up to the fifth order and a comparison with exact solutions are set out in Table 1. The approximate eigenvalues always exceed the exact ones and monotonously tend to the latter with an increasing order n. This trend is observed for all one- and multi-dimensional bodies of simple and complex shapes.

Let

$$\phi_1(Fo) = T_0, \quad \phi_2(Fo) = \phi(Fo), \quad q_v(\rho, Fo) = 0, \quad \Gamma = 0$$

then the truncated system of equations (8) will yield

$$\bar{a}_1(p) = \frac{[T_0 - p\bar{\phi}(p)]N(Bi_1, Bi_2)}{p + A(Bi_1, Bi_2)} \tag{15}$$

where

$$A(Bi_1, Bi_2) = 5\omega[Bi_1 Bi_2 + 4(Bi_1 + Bi_2) + 12] \times (Bi_1 + Bi_2 + Bi_1 Bi_2)$$

$$N(Bi_1, Bi_2) = 2.5\omega Bi_2[Bi_1^2 Bi_2 + 5Bi_1 \times (Bi_1 + Bi_2) + 8Bi_2 + 20Bi_1 + 24]$$

$$\omega^{-1} = Bi_1^2 Bi_2^2 + 24Bi_1 Bi_2 + 7Bi_1 Bi_2(Bi_1 + Bi_2) + 16(Bi_1^2 + Bi_2^2) + 80(Bi_1 + Bi_2) + 120. \tag{16}$$

The relative excess temperature in a plane wall in the case of an exponential variation in the temperature of the surroundings

$$\phi(Fo) = T_0 + (T_c - T_0)[1 - \exp(-Pd\,Fo)] \tag{17}$$

will be written as

$$\theta(\rho, Fo) = \frac{T(\rho, Fo) - T_0}{T_c - T_0} = \frac{Bi_2[1 - \exp(-Pd\,Fo)](1 + Bi_1\rho)}{Bi_1 + Bi_2 + Bi_1 Bi_2} + \frac{N\,Pd}{A - Pd} \times \{\exp[-A(Bi_1, Bi_2)Fo] - \exp(-Pd\,Fo)\} \times \left[\frac{(2+Bi_2)(1+Bi_1\rho)}{Bi_1 + Bi_2 + Bi_1 Bi_2} - \rho^2\right]. \tag{18}$$

When $Pd \to \infty$, equation (18) will give in the limit the solution for a jump-wise variation in the surrounding medium temperature $\phi(Fo) = T_c > T_0$. The variation of θ predicted by equation (18) at $Pd = \infty$, $Bi_1 = 1$, $Bi_2 = 2$ coincides completely with exact values at the points $\rho = 0$, 0.5, and 1 when $Fo \geqslant 0.05$ [2]. The convergence improves at the finite number Pd.

Table 1. Computation of the eigenvalues for the problem of heat conduction of a cylinder and comparison with exact values

k	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	Exact values
1	6	5.7841	5.7832	5.7831	5.7831	5.7831
2		36.8825	30.7121	30.4716	30.4715	30.4715
3			113.5047	78.2398	74.8960	74.8865
4				269.3183	161.1534	139.0395
5					450.0123	222.9318

The results of comparison of $A(Bi_1, Bi_2)$ by equation (16) with exact values of μ_1^2 from [4]

$$\operatorname{ctg}\mu = \frac{\mu^2 - Bi_1 Bi_2}{\mu(Bi_1 + Bi_2)} \tag{19}$$

are given in Table 2.

It will be assumed for problems (1)–(3) that

$$\phi_1(Fo) = \phi(Fo), \quad \phi_2(Fo) = T_0, \quad q_v = 0, \quad \Gamma = 1$$

then the coefficient-transform $\bar{a}_1(p)$ can also be determined from equation (15) with the only difference that the quantities A and N for the round tube wall are equal to

$$A(Bi_1, Bi_2, k) = 10\omega[Bi_1^2 Bi_2^2(k+1) + Bi_1^2 Bi_2(6k+4) + Bi_2^2 Bi_1(4k+6) + 20(k+1)Bi_1 Bi_2 + Bi_1^2(2k+6) + Bi_2^2(6k+2) + 24(Bi_1 + Bi_2)]$$

$$N(Bi_1, Bi_2, k) = \omega[Bi_1^2 Bi_2^2(2k+3) + Bi_1 Bi_2(12k+8) + Bi_1^2(7k+18) + Bi_1(25k+15) + Bi_2(35k+65) + 60(k+1)]$$

$$\omega^{-1} = Bi_1^2 Bi_2^2(k+1) + Bi_1^2 Bi_2(8k+6) + Bi_1 Bi_2^2(6k+8) + 42 Bi_1 Bi_2(k+1) + Bi_1^2(22k+10) + Bi_2^2(10k+22) + 10Bi_1(10k+6) + 10Bi_2(6k+10) + 120(k+1). \tag{20}$$

Equations (15), (16) and (20) allow one to find the solutions for a plate and for a tube wall at any forms of the entry function $\phi(Fo)$. According to the present method, equation (20) yields, with a slight excess, the square of the first root in the characteristic heat conduction equation for a hollow cylinder under asymmetric third-kind boundary conditions when the surfaces are exposed to media with different heat transfer coefficients a_1 and a_2. To the best of the present author's knowledge, the form and solution of such an equation are not available in the literature. However, under the third- and second-kind boundary conditions ($Bi_1 = Bi \neq 0$, $Bi_2 = 0$) equation (20) gives

$$A(Bi, k) = \frac{10Bi[Bi(k+3)(k+1) + 12]}{(k-1)[Bi^2(k-1)^2(11k+5) + 10Bi(k+1)(5k+3) + 60(k+1)]} \tag{21}$$

which coincides well with μ_1^2 of the equation [5]

$$\frac{J_1(\mu) + Bi/\mu J_0(\mu)}{Y_1(\mu) + Bi/\mu Y_0(\mu)} \cdot \frac{Y_1(k\mu)}{J_1(k\mu)} = 1. \tag{22}$$

(2) The representation of temperature fields by simple and rather accurate expressions has made it possible to find effective analytical solutions for temperature stresses in elastic deformations occurring due to non-stationary temperature gradients in a body exposed to different external thermal effects, including a thermal shock [2, 6]. Let the temperature, found from equation (12) under boundary conditions (17) and $q_v = 0$, be substituted into the equations for shear stresses inside a cylinder and a sphere [7]. This will

Table 2. Comparison of the value of $A(Bi_1, Bi_2)$ (upper line) with μ_1^2 of equation (19) according to the data of ref. [4] (lower line)

	Bi_1						
Bi_2	0.1	0.5	5	10	20	50	∞
0.1	0.1962	0.5388	1.8906	2.2201	2.4243	2.5664	2.6667
	0.1967	0.5384	1.9048	2.2436	2.4551	2.5975	2.6992
0.5	0.5384	0.9208	2.4953	2.8880	3.1347	3.3002	3.4115
	0.5388	0.9216	2.4775	2.8595	3.0976	3.2555	3.3745
5	1.9048	2.4926	5.2380	6.0213	6.5239	6.8641	7.1353
	1.8906	2.4775	5.2167	5.9878	6.4719	6.8069	7.0438
10	2.2437	2.8881	6.0213	6.9566	7.5664	7.9830	8.2782
	2.2201	2.8595	5.9878	6.9064	7.4966	7.9017	8.1967
20	2.4551	3.1347	6.5239	7.5664	8.2539	8.7264	9.0462
	2.4243	3.0976	6.4719	7.4966	8.1682	8.6260	8.9580
50	2.5975	3.3003	6.8641	7.9830	8.7269	9.2410	9.5664
	2.5664	3.2555	6.8069	7.9017	8.6260	9.1264	9.4864

yield

$$\sigma_\phi(\rho, Fo, \Gamma) = -\frac{(T_c - T_0)\beta E}{1-\nu}\frac{Pd\,A(Bi,\Gamma)[\Gamma-(\Gamma+2)\rho^2]}{2(\Gamma+1)(\Gamma+3)(A-Pd)} \times \{\exp[-A(Bi,\Gamma)Fo] - \exp(-Pd\,Fo)\}. \quad (23)$$

The variation of

$$\tilde{\sigma}_\phi = \frac{\sigma_\phi(\rho, Fo)(1-\nu)}{\beta E(T_c - T_0)}$$

on the cylinder and sphere surface at $Pd = 2$, $Bi = 1$, 10, and ∞ and comparison with exact solutions are shown in Fig. 1. The maximum shear stress is attained on the surface of a body. It is seen from the plots that a fixed value of Pd such a value of Fo^* is achieved at which this stress attains its critical value σ_ϕ^*. The differentiation of equation (23) with respect to Fo at $\rho = 1$ yields

$$Fo^* = [\ln A(Bi,\Gamma) - \ln Pd]\cdot[A(Bi,\Gamma) - Pd]^{-1} \quad (24)$$

$$\tilde{\sigma}_\phi^* = \frac{\sigma_\phi^*(1-\nu)}{\beta E(T_c-T_0)} = \frac{A(Bi,\Gamma)}{(\Gamma+1)(\Gamma+3)} \times \left[\frac{A(Bi,\Gamma)}{Pd}\right]^{A(Bi,\Gamma)/(Pd - A(Bi,\Gamma))} \quad (25)$$

At fixed numbers Bi and Pd the above equations give the time Fo^* and the value of the maximum critical stress σ_ϕ^* in the units and components in the form of a plate, cylinder and a sphere in the case of emergency shut-downs of atomic and thermal power stations [8]. It should be noted that these equations give a very high accuracy of predictions. Analogous equations were also obtained for the walls of a pipeline [6].

(3) Let the operator-corollary dependence of temperature $T(\rho, Fo)$ on the causes of thermal perturbation $\phi_1(Fo)$, $\phi_2(Fo)$, $q_v(\rho, Fo)$ be written, according to the proposed method, as an approximate solution of the heat conduction problem in the form

$$T_n(\rho, Fo) \doteq H_n[\phi_1(Fo), \phi_2(Fo), q_v(\rho, Fo), \rho, Fo, Bi, \Gamma]. \quad (26)$$

The representation of the operator H_n for one arbitrary and the rest fixed entry functions of the causes by a simple and accurate enough functional relation makes it possible to develop an efficient analytical method for solving inverse heat conduction problems (IHP). Below, an IHP is presented which is related to the problem of controlling the internal source of heat generation by means of the set admissible temperatures on the body surface. Let $q_v(\rho, Fo) = q_v(\rho)w(Fo)$, where $q_v(Fo)$ is the known function and $w(Fo)$ is the control function. The temperature field inside of a fuel element having the form of a cylinder at $q_v(\rho) = q_v(1+\delta\rho^2)$, $q_v = \text{const.}$ and under third-kind boundary conditions, when the ambient temperature is kept constant and equal to T_0, is found to first approximation in the form

$$T_1(\rho, Fo) = H_1[q_v(\rho, Fo), \rho, Fo] = T_0 + \frac{q_v R^2 A(Bi,\delta)}{\lambda}\int_0^{Fo}\exp[-A(Bi,\delta)(Fo-\tau)]w(\tau)\,d\tau \times \left[\frac{8+4\delta+Bi(4+\delta)}{Bi} - 4\rho^2 - \delta\rho^4\right] \quad (27)$$

where

$$A(Bi,\delta) = \frac{10Bi[Bi(3\delta^2+16\delta+24)+24(\delta+2)^2]}{Bi^2(4\delta^2+25\delta+40)+40Bi(\delta^2+5\delta+6)+120(\delta+2)^2}. \quad (28)$$

By virtue of the fact that the optimal first basic function has been selected, solution (27) in the class of functions $w(Fo)$ that satisfy the condition

$$\lim_{Fo\to\infty} w(Fo) = \lim_{p\to 0} p\bar{w}(p) = 1$$

will rather rapidly approach the exact limiting solution with an increase of Fo. Assume that $T(1, Fo) - T_0 = \Phi(Fo)$ is the admissible temperature on the body surface. Then equation (27) will yield

$$\Phi(Fo) = \frac{4q_v R^2(\delta+2)A(Bi,\delta)}{\lambda\,Bi} \times \int_0^{Fo}\exp[-A(Bi,\delta)(Fo-\tau)]w(\tau)\,d\tau$$

where

$$w(Fo) = \frac{\lambda\,Bi}{4q_v R^2(\delta+2)}\left[\frac{1}{A(Bi,\delta)}\cdot\frac{d\Phi}{dFo} + \Phi(Fo)\right]. \quad (29)$$

The inverse heat conduction problems associated with the heat flux recovery on the surface of a plate and tube wall from the temperature curve recorded in time at one point are reduced to the Volterra integral equations with convolution-type kernels the solutions of which are found in the form of simple integrals [9].

(4) In order to preserve the exponential stabilization of the temperature fields in time and along the longitudinal coordinate in the solutions of the internal problems of convective heat transfer for fluid flows in

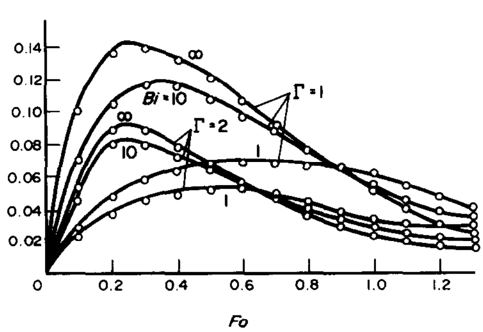

FIG. 1. Variation of shear stresses on the surface of a cylinder and a sphere: ——, calculation by equation (23); ○, exact solution.

straight tubes, it is necessary to carry out twice the Laplace–Karson integral transformation with respect to parabolic variables and then, for the transformed equation of transfer from the rest elliptic coordinates (one or two variables), to construct a discrete analogue with the application of the finite element method to the entire region of the tube cross-section be means of the residual orthogonalization to the basic functions. By taking the entire tube cross-section as a finite element, without breaking it up, it is possible to employ most efficiently the conditions of thermal loading at the boundary when setting up the structure of an approximate analytical solution. This method was used for the solution of unsteady-state problems of heat transfer in a plane channel and round tube [12]. Under constant temperature conditions on the walls and at the entrance to the tube these solutions gave a good coincidence, already in the third approximation, with the solutions of ref. [11] which had been found by the method of characteristics in the sixth and seventh approximations. Note that the method suggested in the present paper finds efficient solutions at any variable thermal loading [2, 12].

The advantage of simultaneous application of the Laplace–Karson double integral transformation and of the projection method to unsteady-state problems of heat transfer is especially evident when it is required to determine the temperature in tubes with non-classical cross-sections and it is necessary to take into account their two-dimensional character and when the temperature fields in the flow of the medium depend on x, y, z and time t.

(5) Now, the solution of a problem will be given which also involves two integral transformations but with different kernels. Determine, in a long cylindrically shaped rod ($0 \leqslant x < \infty$), the redistribution of temperature $T(r, x, t)$ due to the given heat flux to the centred portion of the end surface and thermal insulation of the remainder

$$\left(-\lambda\frac{\partial T}{\partial x}\right)_{x=0} = q(r,t), \quad 0 \leqslant r \leqslant r_0$$
$$\left(-\lambda\frac{\partial T}{\partial x}\right)_{x=0} = 0, \quad r_0 < r \leqslant R. \tag{30}$$

A portion of heat is removed through the side surface ($r = R$, $0 < x < \infty$) into the surrounding medium having a constant temperature T_0. The following notation will be used for the temperature field in dimensionless variables

$$T^*(\rho,\xi,Fo) = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty T(\rho,X,Fo)\cos\xi X\,\mathrm{d}x.$$

Then, after carrying out the Fourier cosine transformation and Laplace transformation over the space ($0 \leqslant X < \infty$, $0 \leqslant Fo < \infty$) taking into account the transition formula

$$\frac{\partial^2 T}{\partial X^2} - \frac{\partial T}{\partial Fo} \doteqdot -(\xi^2+p)\bar{T}^*(\rho,\xi,p) + \sqrt{\left(\frac{2}{\pi}\right)}\left(-\frac{\partial T}{\partial X}\right)_{X=0}$$

the set heat conduction problem will be reduced to the form

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}\bar{T}^*}{\mathrm{d}\rho}\right) - (\xi^2+p)\rho\bar{T}^*(\rho,\xi,p) + \sqrt{\left(\frac{2}{\pi}\right)}\frac{\bar{q}(\rho,p)\rho R}{\lambda} = 0 \tag{31}$$

$$\left\{\frac{\mathrm{d}\bar{T}^*}{\mathrm{d}\rho} + Bi\,\bar{T}^*(\rho,\xi,p)\right\}_{\rho=1} = 0, \quad \left(\frac{\mathrm{d}\bar{T}^*}{\mathrm{d}\rho}\right)_{\rho=0} = 0. \tag{32}$$

Here, without losing the generality of the method, it was assumed that $T_0 = 0$. An approximate solution, which satisfies exactly boundary conditions (32), lies in the set

$$\bar{T}^*_n(\rho,\xi,p) = \sum_{k=1}^{n}\bar{a}^*_k(\xi,p)\psi_k(\rho) \tag{33}$$

where

$$\psi_1(\rho) = \frac{Bi+2}{Bi} - \rho^2,$$
$$\psi_k(\rho) = (1-\rho^2)^2\rho^{2(k-2)}, \quad k \geqslant 2.$$

Assume that

$$q(\rho,Fo) = q(Fo)\phi(\rho), \quad d_j = \int_0^1 \phi(\rho)\psi_j(\rho)\,\mathrm{d}\rho$$

then it can be found from the governing system of equations of type (8) that

$$\bar{a}^*_k(\xi,p) = \sqrt{\left(\frac{2}{\pi}\right)}\frac{\bar{q}(p)R}{\lambda}\cdot\frac{\Delta_k(\omega)}{\Delta(\omega)},$$
$$\omega = \xi^2+p, \quad k = 1,2,\ldots,n. \tag{34}$$

The roots of the basic determinant $\Delta(\omega)$ are always simple and negative. By designating them as $\omega_1 = -A_1^{(n)}(Bi) < 0, \ldots, \omega_n = -A_n^{(n)}(Bi) < 0$, the exact fraction $\Delta_k(\omega)/\Delta(\omega)$ is expanded in a sum over simple poles

$$\frac{\Delta_k(\omega)}{\Delta(\omega)} = \sum_{i=1}^{n}\frac{\Delta_k(\omega_i)}{\Delta'(\omega_i)}\cdot\frac{1}{p+\xi^2+A_i^{(n)}(Bi)}.$$

The inversion of equation (34) yields

$$a_k(X,Fo) = \frac{R}{\lambda\sqrt{\pi}}\sum_{i=1}^{n}\frac{\Delta_k(\omega_i)}{\Delta'(\omega_i)}\int_0^{Fo}\frac{\exp[-A_i^{(n)}(Bi)(Fo-\tau)]\exp[-X^2/4(Fo-\tau)]}{\sqrt{(Fo-\tau)}}q(\tau)\,\mathrm{d}\tau. \tag{35}$$

Performing the transition to inverse transforms in equation (33) by means of equation (35), changing the order of summation over indices k and i and carrying out the apparent collection of terms will yield

$$T_n(\rho, X, Fo) = T_0 + \frac{R}{\lambda\sqrt{\pi}} \sum_{i=1}^{n} \left\{ \int_0^{Fo} \frac{\exp[-A_i^{(n)}(Bi)(Fo-\tau)]\exp[-X^2/4(Fo-\tau)]q(\tau)\,d\tau}{\sqrt{(Fo-\tau)}} \right\} \psi_i^*(\rho, Bi) \quad (36)$$

where

$$\psi_i^*(\rho, Bi) = \sum_{k=1}^{n} b_{ik}\psi_k(\rho), \quad b_{ik} = \frac{\Delta_k(\omega_i)}{\Delta'(\omega_i)}, \quad b = \begin{Vmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{Vmatrix}.$$

Here b is the matrix of transition from the polynomial bases $\{\psi_k(\rho)\}$ to the quasi-orthogonal system of bases $\{\psi_i^*(\rho, Bi)\}$ which approximate with a high accuracy the system of the Bessel orthogonal functions—the eigenfunctions of the elliptical operator of heat absorption under third-kind boundary conditions. System $\{\psi_k^*(\rho, Bi)\}$ and solution (36) were largely calculated for $n = 1, 2, 3, 4$, $Bi = 1, 4, \infty$.

Assume in solution (36) that $Bi = 0$, $\phi(\rho) = 1$, $0 \leqslant \rho \leqslant 1$, then it can be found for an isolated rod of radius R in dimensional coordinates that

$$T(x,t) = T_0 + \frac{\sqrt{a}}{\lambda\sqrt{\pi}} \int_0^t \frac{q(\tau)}{\sqrt{(t-\tau)}} \exp\left[-\frac{x^2}{4a(t-\tau)}\right] d\tau. \quad (37)$$

This expression coincides completely with the exact solution for a semi-infinite medium (e.g. in soil) under the second-kind boundary conditions.

Let $T(x_1, t) - T_0 = F(t)$ be the result of the interpolation of the temperature curve measured at point x_1. When $x_1 = 0$, solution (37) yields

$$q(t) = \frac{\sqrt{(c\gamma\lambda)}}{\sqrt{\pi}} \frac{d}{dt} \int_0^t \frac{F(\tau)\,d\tau}{\sqrt{(t-\tau)}}. \quad (38)$$

When $x_1 \neq 0$, the solution of an IHP becomes more complicated and incorrect. In the case of more stringent limits on $F(t)$ the heat flux on the body surface is recovered on the basis of the temperature 'response' at one inner point $x_1 \neq 0$ in the form

$$q(t) = \sqrt{\left(\frac{c\gamma\lambda}{\pi}\right)} \int_0^t \left\{ F'(\tau) + \frac{x_1^2}{2!a} F''(\tau) - \frac{x_1^4}{4!a^2} F'''(\tau) + \cdots \right\} \frac{d\tau}{\sqrt{(t-\tau)}} + \sqrt{(c\gamma\lambda)} \left\{ \frac{x_1}{\sqrt{a}} F'(t) - \frac{x_1^3}{3!\sqrt{(a^3)}} F''(t) + \frac{1}{5!}\left(\frac{x_1}{\sqrt{a}}\right)^3 F'''(t) + \cdots \right\}. \quad (39)$$

Thus, the application of integral transformations (exact methods) over unilateral variables jointly with the orthogonal projection of a residual (approximate method) over the domain of variation of bilateral elliptic coordinates provides an efficient analytical method for solving heat transfer problems. In this case, a unified algorithm makes it possible to very simply solve a rather complex problem of operational calculus—the recovery of the inverted transform by the available transform.

CONCLUSION

By using integral transformations with different kernels over unilateral parabolic variables in conjunction with the finite element method implemented via the orthogonal projection of a residual to the basic axes of the functional space over the entire range of the rest bilateral elliptic coordinates of the instantaneous point, an effective analytical method has been developed for solving direct and inverse problems of unsteady-state heat conduction, internal problems of heat transfer in tubes and of thermoelastic stresses. The solutions are presented in the form of polynomials in elliptical coordinates the coefficients of which stabilize exponentially along the change in the parabolic variables.

The method provides the solution to the Graetz–Nusselt type unsteady-state generalized problems for tubes with the two-dimensional profile of the clear area (triangle, sector of a circle, ellipse, trapezoid, etc.). Owing to this, the method compares favourably with the other known analytical methods.

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METHODE DE RESOLUTION DES PROBLEMES DIRECTS ET INVERSES DE TRANSFERTS DE CHALEUR VARIABLES

Résumé—On propose une méthode analytique efficace pour la résolution des problèmes directs et inverses de conduction thermique, de contraintes thermoélastiques et de transfert thermique par des transformations intégrales simples ou doubles eu égard aux variables paraboliques unilatérales et en améliorant la méthode des éléments finis par la projection résiduelle orthogonale dans un certain espace fonctionnel, pour le domaine complet de variation des coordonnées elliptiques bilatérales.

EINE METHODE ZUR BERECHNUNG DIREKTER UND INVERSER PROBLEME DES INSTATIONÄREN WÄRMETRANSPORTS

Zusammenfassung—Zur Lösung direkter und inverser Probleme der Wärmeleitung, thermoelastischer Spannungen und des Wärmeübergangs wird eine effektive analytische Methode vorgeschlagen. Zunächst wird eine einfache und doppelte Integral-Transformation mit unsymmetrischen parabolischen Variablen eingeführt. Dann wird die Methode der Finiten Elemente für die orthogonale Residuum-Projektion in einem speziellen Funktionalraum über den ganzen Variationsbereich der symmetrischen elliptischen Koordinaten implementiert.

ОБ ОДНОМ МЕТОДЕ РАСЧЕТА ПРЯМЫХ И ОБРАТНЫХ ЗАДАЧ НЕСТАЦИОНАРНОГО ТЕПЛООБМЕНА

Аннотация—Однократным и двухкратным применением интегральных преобразований по односторонним параболическим переменным и реализацией метода конечных элементов ортогональной проекции невязки в некотором функциональном пространстве по всей области изменения двухсторонних эллиптических координат предложен эффективный аналитический метод решения прямых и обратных задач теплопроводности, термоупругих напряжений и теплообмена.